

# The spin $s$ quantum spin chain and the alternating $s^1, s^2$ spin chain with boundaries

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## Abstract

The spin  $s$  quantum spin chain and the alternating  $s^1, s^2$  ( $s^1 - s^2 = \frac{1}{2}$ ) spin chain with boundaries are considered. The scattering of their excitations with the boundaries via the Bethe ansatz method is studied, and the exact boundary  $S$  matrices are computed in the limit  $s, s^{1,2} \rightarrow \infty$ . Moreover, the connection of these models with the  $SU(2)$  Principal Chiral,  $WZW$  and the  $RSOS$  models is discussed.

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# 1 Introduction and Review

Quantum spin chains are 1D quantum mechanical models of  $N$  microscopic degrees of freedom, namely spins, with the Heisenberg ( $XXX$ ) model solved by Bethe [1], [2], [3] being the prototype. Two types of quantum spin chains exist, known as “closed” (periodic boundary conditions) and “open”. Closed spin chains have been very well studied (see e.g. [2]–[4]), whereas open chains are less widely studied, although of great interest. A very interesting aspect of these models is that in the continuum limit, they correspond to 1+1-dimensional integrable quantum field theories [5]. For example the  $S$  matrix that describes the anisotropic Heisenberg ( $XXZ$ ) model in a certain regime also describes a massive integrable quantum field theory, namely the Thirring model or the sine-Gordon model. Also, the  $S$  matrix that describes the bulk scattering for the spin  $s$   $XXX$  chain coincides with the one of the Wess–Zumino–Witten ( $WZW$ ) model [6]. Finally, open spin chains and the corresponding 2D field theories with boundaries display a rich variety of boundary phenomena which — for integrable models — can be investigated exactly.

During the last years, there has been an increased research interest on integrable models with boundaries, especially after the prototype work of Cherednik and Sklyanin [7], [8]. In particular, Cherednik introduced the reflection equation in order to obtain boundary scattering matrices for theories in half line. In the presence of boundaries in addition to the bulk scattering, the scattering of the particles with the boundaries —described by the boundary  $S$  matrix— should also be considered. The boundary  $S$  matrix satisfies a collection of algebraic constraints namely the boundary Yang–Baxter (reflection) equation [7]. Sklyanin considered this equation in the spin chain framework and he generalized the Quantum Inverse Scattering Method for integrable models with boundaries. Moreover, Ghoshal and Zamolodchikov [9] found solutions of the reflection equation for the sine-Gordon model while De Vega and Gonzalez-Ruiz made similar calculations for the  $XXX$  ( $SU(2)$ ),  $XXZ$  ( $A_1^{(1)}$ ),  $XYZ$  and any  $SU(n)$ ,  $A_{n-1}^{(1)}$  spin chain [10].

In this study the spin  $s$  quantum spin chain and the alternating  $s^1$ ,  $s^2$  chain with boundaries are explored. As a warm up, both models with periodic boundaries conditions are reviewed and their relation with 2D quantum field theory is discussed. To describe the models it is necessary to introduce the basic constructing element, namely the  $R$  matrix, which is a solution of the Yang–Baxter equation [11], [12]

$$R_{12}(\lambda_1 - \lambda_2) R_{13}(\lambda_1) R_{23}(\lambda_2) = R_{23}(\lambda_2) R_{13}(\lambda_1) R_{12}(\lambda_1 - \lambda_2). \quad (1.1)$$

We focus on the special case of the  $R$  matrix which is related to the spin  $s$  repre-

sentation of  $SU(2)$ , obtained by fusion [13],

$$R_{0k} = \begin{pmatrix} w_0 + w_3 S_k^3 & w_- S_k^- \\ w_+ S_k^+ & w_0 - w_3 S_k^3 \end{pmatrix} \quad (1.2)$$

where  $w_0 = \lambda + \frac{i}{2}$ ,  $w_3 = w_\pm = i$  and  $S^3$ ,  $S^\pm$  are the  $SU(2)$  generators in the spin  $s$  representation and act, in general, on a  $2s + 1$  dimensional space  $V = C^{2s+1}$ . The generators satisfy the following commutation relations

$$[S^+, S^-] = 2S^3, \quad [S^3, S^\pm] = \pm S^\pm \quad (1.3)$$

and for e.g.  $s = 1$  they become,

$$S_k^3 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}, \quad S_k^+ = \sqrt{2} \begin{pmatrix} & 1 & \\ & & 1 \end{pmatrix}, \quad S_k^- = \sqrt{2} \begin{pmatrix} 1 & & \\ & & \\ & & 1 \end{pmatrix} \quad (1.4)$$

For  $s = \frac{1}{2}$  we obtain the well known  $XXX$   $R$  matrix,

$$\begin{aligned} R_{12}(\lambda)_{jj,jj} &= (\lambda + i), \\ R_{12}(\lambda)_{jk,jk} &= \lambda, \quad j \neq k, \\ R_{12}(\lambda)_{jk,kj} &= i, \quad j \neq k, \\ &1 \leq j, k \leq 2. \end{aligned} \quad (1.5)$$

## 1.1 The spin $s$ quantum spin chain

This spin  $s$  chain has been intensively studied by several authors in the bulk (see e.g.[13]–[20]). To construct the chain we derive the transfer matrix considering the  $R$  matrix related to the spin  $s$  representation of  $SU(2)$  (1.2). We introduce a mass scale in our system therefore, we derive the transfer matrix with inhomogeneities  $\Omega$ ,

$$t(\lambda) = \text{tr}_0 T_0(\lambda) \quad (1.6)$$

where

$$T_0(\lambda) = R_{02N}^{2s}(\lambda - \Omega) R_{02N-1}^{2s}(\lambda + \Omega) \cdots R_{02}^{2s}(\lambda - \Omega) R_{01}^{2s}(\lambda + \Omega), \quad (1.7)$$

and  $R_{0i}^{2s}$  is given by (1.2) and acts on  $V_0 \otimes V_i$ . The auxiliary space  $V_0$  is a two dimensional space whereas the quantum space  $V_i$   $i = 1, \dots, 2N$  is a  $2s + 1$  dimensional space. After we diagonalize the transfer matrix we find the following eigenvalues [13]

$$\begin{aligned} \Lambda_1^{2s}(\lambda) &= (\lambda - \Omega + is + \frac{i}{2})^N (\lambda + \Omega + is + \frac{i}{2})^N \prod_{j=1}^M \frac{(\lambda - \lambda_j - \frac{i}{2})}{(\lambda - \lambda_j + \frac{i}{2})} \\ &+ (\lambda - \Omega - is + \frac{i}{2})^N (\lambda + \Omega - is + \frac{i}{2})^N \prod_{j=1}^M \frac{(\lambda - \lambda_j + \frac{3i}{2})}{(\lambda - \lambda_j + \frac{i}{2})} \end{aligned} \quad (1.8)$$

The corresponding Bethe ansatz equations are,

$$e_q(\lambda_\alpha - \Omega)^N e_q(\lambda_\alpha + \Omega)^N = - \prod_{\beta=1}^M e_2(\lambda_\alpha - \lambda_\beta), \quad (1.9)$$

where  $q = 2s$  and

$$e_n(\lambda) = \frac{\lambda + \frac{in}{2}}{\lambda - \frac{in}{2}}.$$

The BAE (1.9) are similar to the ones found in [15] for the lattice analogue of the  $SU(2)$  PCM.

In general, if we fuse the auxiliary space as well, we find that the eigenvalues of the transfer matrix are [13],

$$\Lambda_l^{2s}(\lambda) = \sum_{k=0}^l a_k(\lambda - \Omega)^N a_k(\lambda + \Omega)^N \prod_{j=1}^M \frac{(\lambda - \lambda_j - \frac{i}{2})}{(\lambda - \lambda_j + i(k-1) + \frac{i}{2})} \frac{(\lambda - \lambda_j + il + \frac{i}{2})}{(\lambda - \lambda_j + ik + \frac{i}{2})}, \quad (1.10)$$

where

$$a_k(\lambda) = \prod_{m=k}^{l-1} (\lambda - is + \frac{i}{2} + im) \prod_{n=0}^{k-1} (\lambda + is + in + \frac{i}{2}). \quad (1.11)$$

In the special case where  $l = 2s$  and  $\lambda = \pm\Omega + \lambda_0 = \pm\Omega - \frac{i}{2}(2s-1)$  the  $R_{2s}^{2s}$  becomes the permutation operator, therefore we can derive a local Hamiltonian for the system (up to an additive constant)

$$H \propto \frac{i}{2\pi} \frac{d}{d\lambda} \log(t^{2s}(\lambda))|_{\lambda=\Omega+\lambda_0} + \frac{i}{2\pi} \frac{d}{d\lambda} \log(t^{2s}(\lambda))|_{\lambda=-\Omega+\lambda_0}. \quad (1.12)$$

The corresponding eigenvalues follow from (1.10) and (1.12)

$$E = -\frac{1}{2\pi} \sum_{n=1}^2 \sum_{j=1}^M \frac{q}{(\lambda_j + (-)^n \Omega + \frac{iq}{2})(\lambda_j + (-)^n \Omega - \frac{iq}{2})} \quad (1.13)$$

also the momentum and spin are given by

$$P = \frac{1}{2i} \sum_{n=1}^2 \sum_{j=1}^M \log \frac{(\lambda_j + (-)^n \Omega + \frac{iq}{2})}{(\lambda_j + (-)^n \Omega - \frac{iq}{2})} \quad (1.14)$$

$$S^z = qN - M. \quad (1.15)$$

The ground state and the low lying excitations of the model can be studied, in the thermodynamic limit  $N \rightarrow \infty$ . In this limit, the solutions of the Bethe ansatz equations

are given by the so called string hypothesis, i.e. the solutions of (1.9) are grouped into strings of length  $n$  with the same real part and equidistant imaginary parts

$$\lambda_\alpha^{(n,j)} = \lambda_\alpha^n + \frac{i}{2}(n+1-2j), \quad j = 1, 2, \dots, n \quad (1.16)$$

where  $\lambda_\alpha^n$  is real. It is known [13]–[17], that the ground state, i.e. the state with the least energy, is the filled Dirac sea with strings of length  $q = 2s$ . The low lying excitations are holes in the  $q$  sea and also strings of length  $n \neq q$ . One can study the scattering among the low lying excitations of the models and show [13]–[20] that the  $S$  matrix, as  $s \rightarrow \infty$ , coincides with the one of the  $SU(2)$   $PCM$  i.e., it is the  $S_{SU(2)} \otimes S_{SU(2)}$   $S$  matrix [21], [22]. The  $SU(2)$  scattering amplitudes (for the singlet triplet and singlet respectively) are

$$S_0(\lambda) = \exp\left\{-\int_{-\infty}^{\infty} e^{-i\omega\lambda} \frac{e^{-\frac{\omega}{2}}}{2 \cosh(\frac{\omega}{2})} \frac{d\omega}{\omega}\right\}, \quad S_+(\lambda) = \frac{\lambda+i}{\lambda-i} S_0(\lambda). \quad (1.17)$$

In the scaling limit,  $\lambda \ll \omega$ , the excitations become massive relativistic particles [17], with energy and momentum,

$$\epsilon(\lambda) \sim 2e^{-\pi\Omega} \cosh \pi\lambda, \quad p(\lambda) \sim 2e^{-\pi\Omega} \sinh \pi\lambda \quad (1.18)$$

where the momentum  $p$  is defined *mod*  $\pi$  for even number of excitations.

Faddeev and Reshetikhin showed in [17] that for finite  $s$ , inconsistencies in the counting of the states exist. Therefore, the interpretation of the  $S$  matrix as  $S_{SG}(s) \otimes S_{SU(2)}$  somehow fails;  $S_{SG}(s)$  is the sine-Gordon  $S$  matrix [23], with the triplet amplitude being

$$S'_0(\lambda) = \exp\left\{-\int_{-\infty}^{\infty} e^{-i\omega\lambda} \frac{\sinh\left((2s-1)\frac{\omega}{2}\right)}{2 \cosh(\frac{\omega}{2}) \sinh(2s\frac{\omega}{2})} \frac{d\omega}{\omega}\right\}, \quad (1.19)$$

and obviously  $S_{SG}(s \rightarrow \infty) \rightarrow S_{SU(2)}$ . However, Reshetikhin conjectured in [6] that the correct  $S$  matrix for the spin  $s$  chain is the  $S_{RSOS}(s) \otimes S_{SU(2)}$  matrix which coincides with the  $S$  matrix of the  $WZW$  model at level  $k = 2s$  ( $WZW_k$ ) [21], [6], [22]. The  $S_{RSOS}(s)$  is the scattering matrix of the  $RSOS$  model, and the spin  $s$  is related to the restriction parameter  $r$  of the  $RSOS$  model i.e.  $r = 2s+2$ . (for a more detailed analysis see [24], [6]). More specifically, the spin  $s = 1$  chain has a hidden super-symmetry, which is described by the  $RSOS$  part of the  $S$  matrix [6].

A key observation is that, as  $k = 2s \rightarrow \infty$ , the  $S_{RSOS}(s) \otimes S_{SU(2)}$  matrix degenerates to a tensor product of two rational matrices, and it coincides with the  $S$  matrix of the  $SU(2)$   $PCM$  without topological term, found in [15], [17], [20]. The later comment reflects the fact that the perturbed  $WZW_k$  as  $k \rightarrow \infty$ , reduces to the  $PCM$  without topological term ( $S_{RSOS}(s \rightarrow \infty)$  reduces to  $S_{SU(2)}$ ), as described in [25].

## 1.2 The alternating $s^1$ , $s^2$ spin chain

Alternating spin chains have been originally introduced by de Vega and Woyanorovich in [26] and they have been also studied in the bulk by several authors (see e.g. [27]–[31]). We define the transfer matrix of the chain with inhomogeneities,

$$t = \text{tr}_0 T_0(\lambda) \quad (1.20)$$

where

$$T_0(\lambda) = R_{02N}^1(\lambda - \Omega) R_{02N-1}^2(\lambda + \Omega) \cdots R_{02}^1(\lambda - \Omega) R_{01}^2(\lambda + \Omega), \quad (1.21)$$

and  $R^i$  is related to the spin  $s^i$  ( $i = 1, 2$ ) representation (1.2). The eigenvalues of the transfer matrix, after we fuse the auxiliary space as well, are given by (see also [26])

$$\Lambda_l^{(1,2)}(\lambda) = \sum_{k=0}^l a_k^{(1)}(\lambda - \Omega)^N a_k^{(2)}(\lambda + \Omega)^N \prod_{j=1}^M \frac{(\lambda - \lambda_j - \frac{i}{2})}{(\lambda - \lambda_j + i(k-1) + \frac{i}{2})} \frac{(\lambda - \lambda_j + il + \frac{i}{2})}{(\lambda - \lambda_j + ik + \frac{i}{2})}, \quad (1.22)$$

where

$$a_k^{(j)}(\lambda) = \prod_{m=k}^{l-1} (\lambda - is^j + \frac{i}{2} + im) \prod_{n=0}^{k-1} (\lambda + is^j + in + \frac{i}{2}). \quad (1.23)$$

The corresponding Bethe ansatz equations have the form,

$$e_{q^1}(\lambda_\alpha - \Omega)^N e_{q^2}(\lambda_\alpha + \Omega)^{N-} = - \prod_{\beta=1}^M e_2(\lambda_\alpha - \lambda_\beta), \quad (1.24)$$

where  $q^j = 2s^j$  and  $q^1 - q^2 = 1$ . The BEA (1.24) coincide with the ones found by Polyakov and Wiegmann for the lattice analogue of the  $SU(2)$  PCM with  $WZW$  term [32].

Again in the case that  $l = 2s^j$  ( $j = 1, 2$ ) and  $\lambda = \pm\Omega + \lambda_0^j = \pm\Omega - \frac{i}{2}(2s^j - 1)$  the  $R_{2s^j}^{2s^j}$  matrix becomes the permutation operator and hence we obtain a local Hamiltonian,

$$H \propto \frac{i}{2\pi} \frac{d}{d\lambda} \log(t^1(\lambda))|_{\lambda=\Omega+\lambda_0^1} + \frac{i}{2\pi} \frac{d}{d\lambda} \log(t^2(\lambda))|_{\lambda=-\Omega+\lambda_0^2} \quad (1.25)$$

and the corresponding eigenvalues are

$$E = -\frac{1}{2\pi} \sum_{n=1}^2 \sum_{j=1}^M \frac{q^n}{(\lambda_j + (-)^n \Omega + \frac{iq^n}{2})(\lambda_j + (-)^n \Omega - \frac{iq^n}{2})}. \quad (1.26)$$

Moreover, the momentum and spin of a Bethe state are given by

$$P = \frac{1}{2i} \sum_{n=q^1}^{q^2} \sum_{j=1}^M \log \frac{(\lambda_j + (-)^n \Omega + \frac{iq^n}{2})}{(\lambda_j + (-)^n \Omega - \frac{iq^n}{2})} \quad (1.27)$$

$$S^z = (q^1 + q^2)N - M. \quad (1.28)$$

It has been proved [29], [31], that this model has two types of low lying excitations, i.e. holes. The scattering among them was studied, the corresponding  $S$  matrix was computed and it was shown to be the  $S_{SU(2)} \otimes S_{SU(2)}$  plus a non trivial left right scattering  $S_{LR}(\lambda) = \tanh \frac{\pi}{2}(\lambda - \frac{i}{2})$ . The  $S_{SU(2)} \otimes S_{SU(2)}$ ,  $S_{LR}(\lambda)$  is also the massless  $S$  matrix of the  $SU(2)$  principal chiral model with  $WZW$  term at level 1 ( $PCM_1$ ), conjectured by Zamolodchikovs [21]. It has been also proved [31], that in the scaling limit  $\lambda \ll \Omega$ , both excitations obey a massless relativistic dispersion relation, namely

$$\epsilon^1(\lambda) = p^1(\lambda) \sim e^{-\pi\Omega} e^{\pi\lambda}, \quad \epsilon^2(\lambda) = -p^2(\lambda) \sim e^{-\pi\Omega} e^{-\pi\lambda}. \quad (1.29)$$

These are the energy and momentum of the “right” and “left” movers respectively (see e.g. [21]) and the factor  $e^{-\pi\Omega}$  provides a mass scale for the system.

## 2 Spin chains with boundaries

After the brief review on the bulk spin  $s$  and alternating spin chains, we are ready to study these models in the presence of boundaries. For both models we will derive the Bethe ansatz equations and we will compute the exact reflection matrices. To construct the spin chain with boundaries in addition to the  $R$  matrix another constructing element, the  $K$  matrix, is needed. The  $K$  matrix is a solution of the reflection (boundary Yang–Baxter) equation [7],

$$\begin{aligned} & R_{12}(\lambda_1 - \lambda_2) K_1(\lambda_1) R_{21}(\lambda_1 + \lambda_2) K_2(\lambda_2) \\ &= K_2(\lambda_2) R_{12}(\lambda_1 + \lambda_2) K_1(\lambda_1) R_{21}(\lambda_1 - \lambda_2). \end{aligned} \quad (2.1)$$

In what follows we are going to use Sklyanin’s formalism [8] in order to construct both models with boundaries.

### 2.1 The spin $s$ open spin chain

#### 2.1.1 The Bethe ansatz equations

Let us first consider the spin  $s = \frac{g}{2}$  quantum spin chain with boundaries. We define the transfer matrix for the open chain [8]

$$t(\lambda) = \text{tr}_0 K_0^+(\lambda) T_0(\lambda) K_0^-(\lambda) \hat{T}_0(\lambda), \quad (2.2)$$

$T_0(\lambda)$  is the monodromy matrix (1.7) and

$$\hat{T}_0(\lambda) = R_{10}^{2s}(\lambda - \Omega) R_{20}^{2s}(\lambda + \Omega) \cdots R_{2N-10}^{2s}(\lambda - \Omega) R_{2N0}^{2s}(\lambda + \Omega). \quad (2.3)$$

The  $K^-(\lambda) = K(\lambda, \xi^-)$  matrix satisfies the reflection equation (2.1) and the  $K^+(\lambda) = K^+(-\lambda - \rho)^t (\xi^- \rightarrow -\xi^+)$  satisfies,

$$\begin{aligned} & R_{12}(-\lambda_1 + \lambda_2) K_1^+(\lambda_1)^{t_1} R_{21}(-\lambda_1 - \lambda_2 - 2\rho) K_2^+(\lambda_2)^{t_2} \\ &= K_2^+(\lambda_2)^{t_2} R_{12}(-\lambda_1 - \lambda_2 - 2\rho) K_1^+(\lambda_1)^{t_1} R_{21}(-\lambda_1 + \lambda_2), \end{aligned} \quad (2.4)$$

where  $\rho = i$ . We choose the diagonal solution of the reflection equation [9], [10], namely

$$K(\lambda) = \text{diag}(-\lambda + i\xi, \quad \lambda + i\xi). \quad (2.5)$$

Note that in the presence of boundaries we have to fuse the  $K$  matrix as well, and we use the following fusion hierarchy for the transfer matrix [33], [34]

$$\begin{aligned} t^j(\lambda) &= \tilde{\zeta}_{2j-1}(2\lambda + (2j-1)i) \left[ t^{j-\frac{1}{2}}(\lambda) t^{\frac{1}{2}}(\lambda + (2j-1)i) \right. \\ &\quad \left. - \frac{\Delta(\lambda + (2j-2)i) \tilde{\zeta}_{2j-2}(2\lambda + (2j-2)i)}{\zeta(2\lambda + 2i(2j-1))} t^{j-1}(\lambda) \right] \end{aligned} \quad (2.6)$$

where

$$\Delta(\lambda) = \Delta[K^+(\lambda)] \delta[T(\lambda)] \Delta[K^-(\lambda)] \delta[\hat{T}(\lambda)], \quad (2.7)$$

and the quantum determinants are [8], [13], [33],

$$\begin{aligned} \delta[T(\lambda)] &= \delta[\hat{T}(\lambda)] = \hat{\zeta}(\lambda + \Omega + i)^N \hat{\zeta}(\lambda - \Omega + i)^N, \\ \Delta[K^-(\lambda)] &= g(2\lambda + \rho) l(\lambda, \xi^-), \quad \Delta[K^+(\lambda)] = g(-2\lambda - 3\rho) l(\lambda, \xi^+) \end{aligned} \quad (2.8)$$

moreover

$$\begin{aligned} \zeta(\lambda) &= (\lambda + i)(-\lambda + i), \quad \hat{\zeta}(\lambda) = (\lambda + is + \frac{i}{2})(-\lambda + is + \frac{i}{2}), \\ g(\lambda) &= -\lambda + i, \quad l(\lambda, \xi) = (\lambda + i + \xi)(\lambda + i - \xi) \end{aligned} \quad (2.9)$$

and

$$\tilde{\zeta}_j(\lambda) = \prod_{k=1}^j \zeta(\lambda + ik), \quad \tilde{\zeta}_0(\lambda) = 1. \quad (2.10)$$

Then the eigenvalues of the transfer matrix, after we fuse the auxiliary space, are given by

$$\begin{aligned} \Lambda_l^{2s}(\lambda) &= \sum_{k=0}^l h_k(\lambda) f_k(\lambda) a_k(\lambda - \Omega)^{2N} a_k(\lambda + \Omega)^{2N} \\ &\quad \prod_{j=1}^M \frac{(\lambda - \lambda_j - \frac{i}{2})}{(\lambda - \lambda_j + i(k-1) + \frac{i}{2})} \frac{(\lambda - \lambda_j + il + \frac{i}{2})}{(\lambda - \lambda_j + ik + \frac{i}{2})} \\ &\quad \prod_{j=1}^M \frac{(\lambda + \lambda_j - \frac{i}{2})}{(\lambda + \lambda_j + i(k-1) + \frac{i}{2})} \frac{(\lambda + \lambda_j + il + \frac{i}{2})}{(\lambda + \lambda_j + ik + \frac{i}{2})} \end{aligned} \quad (2.11)$$



where  $a_k$  have been already defined in (1.11) and

$$f_k(\lambda) = f_k^+(\lambda)f_k^-(\lambda) \quad (2.12)$$

where

$$f_k^\pm(\lambda) = \prod_{m=k}^{l-1} (\lambda + i\xi^\pm + im + i) \prod_{n=0}^{k-1} (-\lambda + i\xi^\pm - in) \quad (2.13)$$

$f_k^\pm(\lambda)$  correspond to the left (+) and right (-) boundary.  $h_k(\lambda)$  are derived by the fusion hierarchy (2.6) and for e.g.  $l = 1$ ,  $l = 2$  they are respectively <sup>2</sup>

$$h_k(\lambda) = \frac{1}{2\lambda + i} \prod_{m=k}^{l-1} (2\lambda + 2im) \prod_{n=0}^{k-1} (2\lambda + 2i + 2in) \quad (2.15)$$

and

$$h_k(\lambda) = \prod_{m=k}^{l-1} (2\lambda + 2im) \prod_{n=0}^{k-1} (2\lambda + 2i + 2in). \quad (2.16)$$

From the analyticity of the eigenvalues we obtain the Bethe ansatz equations

$$\begin{aligned} & e_{x^+}^{-1}(\lambda_\alpha) e_{x^-}^{-1}(\lambda_\alpha) e_q(\lambda_\alpha - \Omega)^{2N} e_q(\lambda_\alpha + \Omega)^{2N} e_1(\lambda_\alpha) \\ &= - \prod_{\beta=1}^M e_2(\lambda_\alpha - \lambda_\beta) e_2(\lambda_\alpha + \lambda_\beta), \end{aligned} \quad (2.17)$$

where  $x^\pm = 2\xi^\pm + 1$ . Recall that  $R_{2s}^{2s}(\lambda_0)$  is the permutation operator, therefore we obtain a local Hamiltonian

$$H \propto \frac{i}{2\pi} \frac{d}{d\lambda} \log(t^{2s}(\lambda))|_{\lambda=\pm\Omega+\lambda_0} \quad (2.18)$$

and the corresponding energy eigenvalues

$$E = -\frac{1}{2\pi} \sum_{n=1}^2 \sum_{j=1}^M \frac{q}{(\lambda_j + (-)^n \Omega + \frac{iq}{2})(\lambda_j + (-)^n \Omega - \frac{iq}{2})}. \quad (2.19)$$

### 2.1.2 The boundary $S$ matrix

Now we can study the scattering of the excitations with the boundaries and determine the boundary  $S$  matrix. Before we do that it is necessary to determine what is the ground state and the low lying excitations for the model. The ground state, as in the

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<sup>2</sup>For general  $l$

$$h_k(\lambda) \propto \prod_{m=k}^{l-1} (2\lambda + 2im) \prod_{n=0}^{k-1} (2\lambda + 2i + 2in). \quad (2.14)$$

bulk, is the filled Dirac sea with strings of length  $q = 2s$ . The Bethe ansatz equations for the ground state become

$$\begin{aligned} & X_{qx^+}^{-1}(\lambda_\alpha) X_{qx^-}^{-1}(\lambda_\alpha) X_{qq}(\lambda_\alpha - \Omega)^{2N} X_{qq}(\lambda_\alpha + \Omega)^{2N} e_q(\lambda_\alpha) \\ &= \prod_{\beta=1}^{M_j} E_{qq}(\lambda_\alpha - \lambda_\beta) E_{qq}(\lambda_\alpha + \lambda_\beta). \end{aligned} \quad (2.20)$$

Let us consider  $x^\pm \geq q - 1$ , and define

$$\begin{aligned} X_{nm}(\lambda) &= e_{|n-m+1|}(\lambda) e_{|n-m+3|}(\lambda) \dots e_{(n+m-3)}(\lambda) e_{(n+m-1)}(\lambda) \\ E_{nm}(\lambda) &= e_{|n-m|}(\lambda) e_{|n-m+2|}^2(\lambda) \dots e_{(n+m-2)}^2(\lambda) e_{(n+m)}(\lambda). \end{aligned} \quad (2.21)$$

For the next we need the following notations

$$a_n(\lambda) = \frac{1}{2\pi} \frac{d}{d\lambda} i \log e_n(\lambda) \quad (2.22)$$

and the Fourier transform of  $a_n$ , is given by

$$\hat{a}_n(\omega) = e^{-\frac{n\omega}{2}}. \quad (2.23)$$

We also need the following expressions

$$\left( Z_{nm}(\lambda), A_{nm}(\lambda) \right) = \frac{1}{2\pi} \frac{d}{d\lambda} i \log \left( X_{nm}(\lambda), E_{nm}(\lambda) \right), \quad (2.24)$$

where their Fourier transforms are

$$\hat{Z}_{nm}(\omega) = \frac{e^{-\frac{\max(n,m)\omega}{2}} \sinh\left(\left(\min(n,m)\right)\frac{\omega}{2}\right)}{\sinh\left(\frac{\omega}{2}\right)}, \quad (2.25)$$

$$\hat{A}_{nm}(\omega) = 2 \coth\left(\frac{\omega}{2}\right) e^{-\frac{\max(n,m)\omega}{2}} \sinh\left(\left(\min(n,m)\right)\frac{\omega}{2}\right) - \delta_{nm}. \quad (2.26)$$

Finally, the energy (2.19) takes the form

$$E = - \sum_{n=1}^2 \sum_{\alpha=1}^M Z_{qq}(\lambda_\alpha + (-)^n \Omega). \quad (2.27)$$

The density for the ground state is given by

$$\begin{aligned} \sigma_0(\lambda) &= Z_{qq}(\lambda + \Omega) + Z_{qq}(\lambda - \Omega) - (A_{qq} * \sigma_0)(\lambda) \\ &+ \frac{1}{L} \left( -Z_{qx^-}(\lambda) - Z_{qx^+}(\lambda) + a_q(\lambda) + A_{qq}(\lambda) \right) \end{aligned} \quad (2.28)$$

where  $L = 2N$  is the length of the chain and  $*$  denotes the convolution. The solution of the last equation is

$$\sigma_0(\lambda) = \epsilon(\lambda) + \frac{1}{L} \left( r(\lambda, \xi^+) + r(\lambda, \xi^-) + Q(\lambda) \right) \quad (2.29)$$

where

$$\hat{Q}(\omega) = \frac{\hat{a}_q(\omega)}{1 + \hat{A}_{qq}(\omega)} + \hat{K}_1(\omega), \quad \hat{r}(\omega, \xi^\pm) = -\frac{\hat{Z}_{qx^\pm}(\omega)}{1 + \hat{A}_{qq}(\omega)}, \quad \hat{K}_1(\omega) = \frac{\hat{A}_{qq}(\omega)}{1 + \hat{A}_{qq}(\omega)}, \quad (2.30)$$

and

$$\epsilon(\lambda) = \sum_{i=1}^2 s(\lambda - (-)^i \Omega), \quad \hat{s}(\omega) = \frac{\hat{Z}_{qq}(\omega)}{1 + \hat{A}_{qq}(\omega)}. \quad (2.31)$$

We can write the above Fourier transforms in terms of trigonometric functions using the definitions of  $\hat{a}_q$  and  $\hat{A}_{qq}$ ,

$$\hat{K}_1(\omega) = \frac{e^{\frac{(q-1)\omega}{2}} - e^{-\frac{q\omega}{2}} \cosh(\frac{\omega}{2})}{2 \cosh(\frac{\omega}{2}) \sinh(\frac{q\omega}{2})}, \quad \hat{Q}(\omega) = \hat{K}_1(\omega) + \frac{1}{2 \coth(\frac{\omega}{2}) \sinh(\frac{q\omega}{2})} \quad (2.32)$$

and for  $x^\pm \geq q - 1$

$$\hat{r}(\omega, \xi^\pm) = -\frac{e^{-(x^\pm - q)\frac{\omega}{2}}}{2 \cosh(\frac{\omega}{2})} \quad (2.33)$$

moreover,

$$\hat{s}(\omega) = \frac{1}{2 \cosh(\frac{\omega}{2})}, \quad \epsilon(\lambda) = \sum_{i=1}^2 \frac{1}{2 \cosh(\pi(\lambda - (-)^i \Omega))}. \quad (2.34)$$

Let us consider the state with  $\nu$  holes in the  $q$  sea, where  $\nu$  an even number. Then, in the thermodynamic limit we obtain the density of the state form the Bethe ansatz equation, namely

$$\sigma(\lambda) = \sigma_0(\lambda) + \frac{1}{L} \sum_{i=1}^{\nu} (K_1(\lambda - \lambda_i) + K_1(\lambda + \lambda_i)). \quad (2.35)$$

The energy of the state with  $\nu$  holes in the  $q$  sea (2.27) is given by

$$E = E_0 + \sum_{\alpha=1}^{\nu} \epsilon(\lambda_\alpha), \quad (2.36)$$

where  $E_0$  is the energy of the ground state and  $\epsilon(\lambda)$  is the energy of the hole in  $q$  sea. Finally, we compute the spin of the holes, and we can see from (1.15) that the spin of a hole in  $q$  sea is (1.15)  $s = \frac{f}{2}$ , where  $f$  is an overall factor (see also [13]). We consider the spin of the hole to be  $\frac{1}{2}$  for what follows. We conclude that the hole in the  $q$  sea is a particle like excitation with energy  $\epsilon$ , momentum  $p$  ( $\epsilon(\lambda) = \frac{1}{\pi} \frac{d}{d\lambda} p(\lambda)$ )

$$\epsilon(\lambda) = \sum_{n=1}^2 \frac{1}{2 \cosh(\pi\lambda - (-)^n \Omega)}, \quad p(\lambda) = \pm \frac{\pi}{4} + \sum_{n=1}^2 \frac{1}{2} \tan^{-1}(\sinh(\pi\lambda - (-)^n \Omega)) \quad (2.37)$$

and spin  $s = \frac{1}{2}$ . We can easily check that in the scaling limit where  $\lambda \ll \Omega$  the energy and momentum become from (2.37),

$$\epsilon(\lambda) \sim 2e^{-\pi\Omega} \cosh(\pi\lambda), \quad p(\lambda) \sim 2e^{-\pi\Omega} \sinh(\pi\lambda). \quad (2.38)$$

Note that the excitations in the scaling limit, as in the bulk case satisfy a massive relativistic dispersion relation.

Having studied the excitations of the model we are ready to compute the complete boundary  $S$ -matrix. To do so we follow the formulation developed by Korepin, and later by Andrei and Destri [35], [36]. First we have to consider the so called quantization condition.

$$(e^{2iLp}S - 1)|\tilde{\lambda}_1, \tilde{\lambda}_2\rangle = 0 \quad (2.39)$$

where  $p$  is the momentum of the particle, the hole in our case. For the case of  $\nu$  (even) holes in  $q$  sea we compare the integrated density (2.35) with the quantization condition (2.39). Having also in mind that,

$$\epsilon(\lambda) = \frac{1}{\pi} \frac{d}{d\lambda} p(\lambda) \quad (2.40)$$

we end up with the following expression for the boundary scattering amplitudes (the boundary  $S$  matrix will have the form  $S^\pm = \text{diag}(\alpha^\pm, \beta^\pm)$ ),

$$\alpha^+ \alpha^- = \exp\left\{2\pi i L \int_0^{\tilde{\lambda}_1} d\lambda (\sigma(\lambda) - \epsilon(\lambda))\right\} \quad (2.41)$$

and

$$\alpha^\pm(\lambda, \xi^\pm) = f(\lambda) k_0(\lambda) k_0(\lambda) k_1(\lambda, \xi^\pm) \quad (2.42)$$

where

$$k_0(\tilde{\lambda}_1) = \exp\left\{2\pi i \int_0^{\tilde{\lambda}_1} \frac{1}{4} \left(\sum_{i=1}^{\nu} (K_1(\lambda - \tilde{\lambda}_i) + K_1(\lambda + \tilde{\lambda}_i)) + Q(\lambda)\right) d\lambda\right\} \quad (2.43)$$

the  $x$  dependent part is

$$k_1(\tilde{\lambda}_1, \xi^\pm) = \exp\left\{2\pi i \int_0^{\tilde{\lambda}_1} r(\lambda, \xi^\pm) d\lambda\right\}. \quad (2.44)$$

We are interested in the limit that  $s \rightarrow \infty$ , in this limit we can easily see from (2.30) (see also [16]) that

$$\hat{Q}(\omega) \rightarrow \hat{K}_1(\omega), \quad \hat{K}_1(\omega) \rightarrow \frac{e^{-\frac{\omega}{2}}}{\cosh \frac{\omega}{2}}, \quad \hat{r}(\omega, \xi^\pm) = -\frac{e^{-(x^\pm - q)\frac{\omega}{2}}}{2 \cosh(\frac{\omega}{2})} \quad (2.45)$$

with  $x^\pm - q$  to be a fixed number as  $q \rightarrow \infty$  and  $f(\lambda)$  is an overall CDD factor given by

$$f(\lambda) = \exp\left\{\int_{-\infty}^{\infty} e^{-i\omega\lambda} \frac{1}{2 \cosh(\frac{\omega}{2})} \frac{d\omega}{\omega}\right\} = \tanh \frac{\pi}{2} \left(\lambda + \frac{i}{2}\right). \quad (2.46)$$

Using the above Fourier transforms we end up with the following expressions,

$$\begin{aligned} k_0(\lambda) &= \exp\left\{2 \int_0^\infty \frac{d\omega}{\omega} \sinh(2i\omega\lambda) \frac{\sinh(\frac{3\omega}{2}) e^{-\frac{\omega}{2}}}{\sinh(2\omega)}\right\}, \\ k_1(\lambda, \tilde{\xi}^\pm) &= \exp\left\{- \int_0^\infty \frac{d\omega}{\omega} \sinh(2i\omega\lambda) \frac{e^{-2\tilde{\xi}^\pm\omega}}{\cosh(\omega)}\right\}, \end{aligned} \quad (2.47)$$

where  $2\tilde{\xi}^\pm = 2\xi^\pm - 2s + 1$  is the renormalized boundary parameter and  $k_0(\lambda)$ ,  $k_1(\lambda, \xi^\pm)$  are the  $\xi$  independent and the  $\xi$  dependent part, respectively of the usual  $XXX$  ( $SU(2)$ ) reflection matrix (see e.g. [37]). We notice from (2.42) that there are two copies of the  $\xi$  independent part, whereas just one copy of the  $\xi$  dependent part exists. Recall, that we considered the diagonal  $K$  matrix (2.5) in order to construct our chain, therefore we need to determine the other element of the boundary matrix. To do so we exploit the “duality” symmetry [38] of the transfer matrix for  $\xi^\pm \rightarrow -\xi^\pm$ . Hence, the other element is given by

$$\beta^\pm(\lambda, \tilde{\xi}^\pm) = e_{2\tilde{\xi}^\pm - 1}(\lambda) \alpha^\pm(\lambda, \tilde{\xi}^\pm), \quad (2.48)$$

the ratio  $\frac{\beta^\pm(\lambda, \tilde{\xi}^\pm)}{\alpha^\pm(\lambda, \tilde{\xi}^\pm)}$  is the same as in the  $XXX$  model [37] but with a renormalized boundary parameter.

The above equations (2.42)–(2.48), are simply combined to give the boundary  $S$  matrix as the tensor product of two rational ( $XXX$ ) boundary  $S$  matrices [37] (up to an overall CDD factor) i.e.

$$S(\lambda) = f(\lambda) S_{SU(2)}(\lambda, \tilde{\xi}_1^\pm) \otimes S_{SU(2)}(\lambda, \tilde{\xi}_2^\pm), \quad (2.49)$$

where  $\tilde{\xi}_1^\pm = \tilde{\xi}^\pm$ ,  $\tilde{\xi}_2^\pm \rightarrow \infty$ . This matrix coincides with the one found for the  $SU(2)$   $PCM$  [39] —in our case  $\tilde{\xi}^\pm$  is a free parameter.

However, in analogy with the bulk case, for finite  $s$  the boundary  $S$  matrix is expected to be of the form  $S_{SU(2)}(\lambda, \tilde{\xi}^\pm) \otimes S_{RSOS}(s)$ , where  $S_{RSOS}(s)$  is the boundary  $S$  matrix of the  $RSOS$  model. The  $S_{RSOS}$  matrix is a solution of the boundary Yang–Baxter equation in the  $RSOS$  representation (see e.g. [40]–[42]). The  $S_{SU(2)}(\lambda, \tilde{\xi}^\pm) \otimes S_{RSOS}(s)$  matrix should also describe the  $WZW_{k=2s}$  boundary scattering and it should presumably reduce to the  $S$  matrix we found (2.49), as  $s \rightarrow \infty$ .

## 2.2 The alternating $s^1$ , $s^2$ open spin chain

### 2.2.1 The Bethe ansatz equations

The corresponding transfer matrix  $t(\lambda)$  for the open alternating chain of  $2N$  sites and  $s^1 = \frac{q^1}{2}$ ,  $s^2 = \frac{q^2}{2}$  spins ( $q^1 - q^2 = 1$ ) is (see also e.g., [26], [8]),

$$t(\lambda) = \text{tr}_0 K_0^+(\lambda) T_0(\lambda) K_0^-(\lambda) \hat{T}_0(\lambda), \quad (2.50)$$

where  $T_0(\lambda)$  is the monodromy matrix defined previously in (1.21) and

$$\hat{T}_0(\lambda) = R_{10}^2(\lambda - \Omega) R_{20}^1(\lambda + \Omega) \cdots R_{2N-10}^2(\lambda - \Omega) R_{2N0}^1(\lambda + \Omega). \quad (2.51)$$

Again here we use the diagonal  $K$  matrix (2.5) and the fusion hierarchy (2.6) with

$$\begin{aligned}\delta[T(\lambda)] &= \hat{\zeta}_1(\lambda - \Omega + i)^N \hat{\zeta}_2(\lambda + \Omega + i)^N, \\ \delta[\hat{T}(\lambda)] &= \hat{\zeta}_1(\lambda + \Omega + i)^N \hat{\zeta}_2(\lambda - \Omega + i)^N\end{aligned}\quad (2.52)$$

and

$$\hat{\zeta}_n(\lambda) = (-\lambda + is^n + \frac{i}{2})(\lambda + is^n + \frac{i}{2}). \quad (2.53)$$

The eigenvalues of the transfer matrix, after we fuse the auxiliary space, are given by

$$\begin{aligned}\Lambda_l^{(1,2)}(\lambda) &= \sum_{k=0}^l h_k(\lambda) f_k(\lambda) a_k^{(1)}(\lambda - \Omega)^N a_k^{(1)}(\lambda + \Omega)^N a_k^{(2)}(\lambda - \Omega)^N a_k^{(2)}(\lambda + \Omega)^N \\ &\quad \prod_{j=1}^M \frac{(\lambda - \lambda_j - \frac{i}{2})}{(\lambda - \lambda_j + i(k-1) + \frac{i}{2})} \frac{(\lambda - \lambda_j + il + \frac{i}{2})}{(\lambda - \lambda_j + ik + \frac{i}{2})} \\ &\quad \prod_{j=1}^M \frac{(\lambda + \lambda_j - \frac{i}{2})}{(\lambda + \lambda_j + i(k-1) + \frac{i}{2})} \frac{(\lambda + \lambda_j + il + \frac{i}{2})}{(\lambda + \lambda_j + ik + \frac{i}{2})}\end{aligned}\quad (2.54)$$

provided that  $\{\lambda_j\}$  satisfy the Bethe ansatz equations

$$\begin{aligned}&\prod_{n=q^1}^{q^2} e_n(\lambda_\alpha - \Omega)^N e_n(\lambda_\alpha + \Omega)^N e_1(\lambda_\alpha) e_{x^+}^{-1}(\lambda_\alpha) e_{x^-}^{-1}(\lambda_\alpha) \\ &= - \prod_{\beta=1}^M e_2(\lambda_\alpha - \lambda_\beta) e_2(\lambda_\alpha + \lambda_\beta).\end{aligned}\quad (2.55)$$

$R_{2sj}^{2sj}(\lambda_0^i)$  is the permutation operator, therefore we can obtain a local Hamiltonian for the open chain

$$H \propto \frac{i}{4\pi} \frac{d}{d\lambda} \log(t^1(\lambda))|_{\lambda=\Omega+\lambda_0^1} + \frac{i}{4\pi} \frac{d}{d\lambda} \log(t^2(\lambda))|_{\lambda=-\Omega+\lambda_0^2} \quad (2.56)$$

and

$$\begin{aligned}E &= -\frac{1}{4\pi} \sum_{n=1}^2 \sum_{j=1}^M \frac{q^n}{(\lambda_j - \Omega + \frac{iq^n}{2})(\lambda_j - \Omega - \frac{iq^n}{2})} \\ &\quad - \frac{1}{4\pi} \sum_{n=1}^2 \sum_{j=1}^M \frac{q^n}{(\lambda_j + \Omega + \frac{iq^n}{2})(\lambda_j + \Omega - \frac{iq^n}{2})}.\end{aligned}\quad (2.57)$$

### 2.2.2 The boundary $S$ matrix

The ground state consists of two filled Dirac seas with strings of length  $q^n = 2s^n$ ,  $n = 1, 2$ . Then the Bethe ansatz equations for the ground state become

$$\begin{aligned}&X_{nx^+}^{-1}(\lambda_\alpha^n) X_{nx^-}^{-1}(\lambda_\alpha^n) \prod_{j=q^1}^{q^2} X_{nj}(\lambda_\alpha^n - \Omega)^N X_{nj}(\lambda_\alpha^n + \Omega)^N e_n(\lambda_\alpha^n) \\ &= \prod_{j=q^1}^{q^2} \prod_{\beta=1}^{M_j} E_{nj}(\lambda_\alpha^n - \lambda_\beta^j) E_{nj}(\lambda_\alpha^n + \lambda_\beta^j)\end{aligned}\quad (2.58)$$

where  $n$  can be  $q^1$ ,  $q^2$  and let us consider for simplicity  $x^\pm \geq q^2$ . Finally, the energy (2.57) takes the form

$$E = -\frac{1}{2} \sum_{i,j=q^1}^{q^2} \sum_{\alpha=1}^{M_j} Z_{ij}(\lambda_\alpha^i + \Omega) - \frac{1}{2} \sum_{i,j=q^1}^{q^2} \sum_{\alpha=1}^{M_j} Z_{ij}(\lambda_\alpha^i - \Omega). \quad (2.59)$$

The density that describes the ground state is given by the following integral equations,

$$\begin{aligned} \sigma_0^n(\lambda) &= \sum_{j=q^1}^{q^2} \frac{1}{2} (Z_{nj}(\lambda - \Omega) + Z_{nj}(\lambda + \Omega)) - \sum_{j=q^1}^{q^2} (A_{nj} * \sigma_0^j)(\lambda) \\ &+ \frac{1}{L} (-Z_{nx^+}(\lambda) - Z_{nx^-}(\lambda) + a_n(\lambda) + \sum_{j=q^1}^{q^2} A_{nj}(\lambda)) \end{aligned} \quad (2.60)$$

where  $n$  can be  $q^1$  or  $q^2$ . The solution of the above integral equation is given by

$$\sigma_0^n(\lambda) = \epsilon^n(\lambda) + \frac{1}{L} (r^n(\lambda, \xi^+) + r^n(\lambda, \xi^-) + Q^n(\lambda)) \quad (2.61)$$

where

$$\begin{aligned} \hat{Q}^n(\omega) &= \sum_{i=q^1}^{q^2} \hat{a}_i(\omega) \hat{R}_{ni}(\omega) + \sum_{i=q^1}^{q^2} \hat{K}_1^{in}(\omega), \\ \hat{r}^n(\omega, \xi^\pm) &= - \sum_{i=q^1}^{q^2} \hat{Z}_{ix^\pm}(\omega) \hat{R}_{ni}(\omega), \quad \hat{K}_1^{nm}(\omega) = \sum_{i=q^1}^{q^2} \hat{A}_{ni}(\omega) \hat{R}_{mi}(\omega) \end{aligned} \quad (2.62)$$

and

$$\epsilon^n(\lambda) = \frac{1}{2} \sum_{i=1}^2 s(\lambda - (-)^i \Omega), \quad (2.63)$$

and its Fourier transform is

$$\hat{s}^n(\omega) = \sum_{i,j=q^1}^{q^2} (\hat{Z}_{ij} \hat{R}_{nj})(\omega). \quad (2.64)$$

Here  $R$  is the inverse of the kernel  $K$  of the system of the linear equations (2.60),

$$\hat{K}_{nm}(\omega) = (1 + \hat{A}_{nm}(\omega)) \delta_{nm} + \hat{A}_{nm}(\omega) (1 - \delta_{nm}) \quad (2.65)$$

$$\hat{R}_{nm}(\omega) = \frac{1}{\det \hat{K}} \sum_{j=q^1}^{q^2} ((1 + \hat{A}_{jj}(\omega)) \delta_{nm} (1 - \delta_{nj}) - \hat{A}_{nm}(\omega) (1 - \delta_{nm})), \quad (2.66)$$

where the determinant of  $K$  is, in terms of trigonometric functions,

$$\det \hat{K} = 4 \coth^2\left(\frac{\omega}{2}\right) e^{-\frac{q^1 \omega}{2}} \sinh\left(q^2 \frac{\omega}{2}\right) \sinh\left(\frac{\omega}{2}\right). \quad (2.67)$$

In particular,  $\hat{K}_1^{nm}$  has the following explicit form in terms of trigonometric functions

$$\begin{aligned} \hat{K}_1^{11}(\omega) &= \frac{e^{-\frac{\omega}{2}}}{2 \cosh(\frac{\omega}{2})}, \quad \hat{K}_1^{22}(\omega) = \frac{\sinh\left((q^1 - 2)\frac{\omega}{2}\right)}{2 \cosh(\frac{\omega}{2}) \sinh\left((q^1 - 1)\frac{\omega}{2}\right)} \\ \hat{K}_1^{12}(\omega) &= \hat{K}_1^{21}(\omega) = \frac{1}{2 \cosh(\frac{\omega}{2})} \end{aligned} \quad (2.68)$$

$$\hat{Q}^1(\omega) = \frac{1 + e^{-\frac{\omega}{2}}}{2 \cosh(\frac{\omega}{2})}, \quad \hat{Q}^2(\omega) = \hat{K}^{22}(\omega) + \frac{1}{2 \cosh(\frac{\omega}{2})} + \frac{\sinh(\frac{\omega}{2})}{2 \cosh(\frac{\omega}{2}) \sinh(\frac{q^2 \omega}{2})}. \quad (2.69)$$

The  $\xi$  dependent part for  $x^\pm \geq q^2$  is

$$\hat{r}^1(\omega, \xi^\pm) = -\frac{e^{-(x^\pm - q^1)\frac{\omega}{2}}}{2 \cosh(\frac{\omega}{2})}, \quad \hat{r}^2(\omega, \xi^\pm) = 0 \quad (2.70)$$

finally,

$$\hat{s}^n(\omega) = \frac{1}{2 \cosh(\frac{\omega}{2})}, \quad \epsilon^n(\lambda) = \sum_{i=1}^2 \frac{1}{4 \cosh(\pi(\lambda + (-)^i \Omega))}. \quad (2.71)$$

Let us consider the state with  $\nu_n$  holes in the  $q^n$  sea, where  $\nu_n$  is an even number. Then in the thermodynamic limit we obtain the density of the state from the Bethe ansatz equations, namely

$$\sigma^n(\lambda) = \sigma_0^n(\lambda) + \frac{1}{L} \sum_{i=1}^{\nu_n} (K_1^{nn}(\lambda - \lambda_i) + K_1^{nn}(\lambda + \lambda_i)) \quad (2.72)$$

the energy of the state with  $\nu_n$  holes in the  $q^n$  seas is given (2.59) by

$$E = E_0 + \sum_{\alpha=1}^{\nu_n} \epsilon^n(\lambda_\alpha^n), \quad (2.73)$$

where  $E_0$  is the energy of the ground state and  $\epsilon^n(\lambda)$  is the energy of the hole in  $q^n$  sea. Finally, we compute the spin of the holes from (1.28), and we can verify that the spin of a hole in the  $q^1$  sea is  $s^1 = \frac{1}{2}$  whereas the spin of a hole in the  $q^2$  sea is  $s^2 = 0$ . We conclude that the hole in the  $q^n$  sea is a particle like excitation with energy  $\epsilon^n$ , momentum  $p^n$  ( $\epsilon^n(\lambda) = \frac{1}{\pi} \frac{d}{d\lambda} p^n(\lambda)$ )

$$\begin{aligned} \epsilon^n(\lambda) &= \sum_{l=1}^2 \frac{1}{4 \cosh(\pi(\lambda - (-)^l \Omega))}, \\ p^n(\lambda) &= \pm \frac{\pi}{4} + \sum_{l=1}^2 \frac{1}{4} \tan^{-1}(\sinh \pi(\lambda - (-)^l \Omega)), \end{aligned} \quad (2.74)$$

and spin  $s^1 = \frac{1}{2}$ ,  $s^2 = 0$ . We can easily check that in the scaling limit,  $\lambda \ll \Omega$  the energy and momentum become from (2.74),

$$\epsilon^n(\lambda) \sim e^{-\pi \Omega} \cosh(\pi \lambda), \quad p^n(\lambda) \sim e^{-\pi \Omega} \sinh(\pi \lambda), \quad (2.75)$$

the factor  $e^{-\pi \Omega}$  provides a mass scale for the system. Note that in the presence of boundaries the excitations, in the scaling limit, satisfy a massive relativistic dispersion relation (2.75) whereas in the bulk case the excitations are massless relativistic particles (1.29). This is a very interesting phenomenon which is presumable related to the type of the boundaries. The boundaries we impose, force a left (right) mover to reflect as a left (right) mover. It is possible that the boundary can reflect a left mover to a



right one and vice versa. This type of boundaries would probably lead to massless excitations in the scaling limit.

Again, we consider the quantization condition [35], [36], in order to compute the exact reflection matrices, namely

$$(e^{2iLp^n} S^n - 1)|\tilde{\lambda}_1, \tilde{\lambda}_2\rangle = 0 \quad (2.76)$$

where  $p^n$  is the momentum of the hole in the  $q^n$  sea. For the case of  $\nu^n$  holes in  $q^n$  sea we compare the integrated density (2.72) with the quantization condition (2.76). Having also in mind that,

$$\epsilon^n(\lambda) = \frac{1}{\pi} \frac{d}{d\lambda} p^n(\lambda) \quad (2.77)$$

we end up with the following expression for the boundary scattering amplitudes ( $S_{\pm}^n = \text{diag}(\alpha_{\pm}^n, \beta_{\pm}^n)$ )

$$\alpha_+^n \alpha_-^n = \exp\left\{2\pi i L \int_0^{\tilde{\lambda}_1} d\lambda (\sigma(\lambda) - \epsilon(\lambda))\right\} \quad (2.78)$$

and

$$\alpha_{\pm}^n(\lambda, \xi^{\pm}) = k_0^n(\lambda) k_1^n(\lambda, \xi^{\pm}) \quad (2.79)$$

where

$$k_0^n(\tilde{\lambda}_1) = \exp\left\{\pi i \int_0^{\tilde{\lambda}_1} \sum_{i=1}^{\nu} (K^{nn}(\lambda - \tilde{\lambda}_i) + K^{nn}(\lambda + \tilde{\lambda}_i)) + Q^n(\lambda) d\lambda\right\} \quad (2.80)$$

the  $x$  dependent part is

$$k_1^n(\tilde{\lambda}_1, \xi^{\pm}) = \exp\left\{2\pi i \int_0^{\tilde{\lambda}_1} r^n(\lambda, \xi^{\pm}) d\lambda\right\}. \quad (2.81)$$

We are interested in the limit that  $q^n \rightarrow \infty$ , in this limit we can easily verify that

$$\hat{Q}^n(\omega) \rightarrow \hat{K}^{nn}(\omega) + \frac{1}{2 \cosh(\frac{\omega}{2})}, \quad \hat{K}^{nn}(\omega) \rightarrow \frac{e^{-\frac{\omega}{2}}}{2 \cosh \frac{\omega}{2}} \quad (2.82)$$

and

$$\hat{r}^1(\omega, \xi^{\pm}) = -\frac{e^{-(x^{\pm}-q^1)\frac{\omega}{2}}}{2 \cosh(\frac{\omega}{2})}, \quad \hat{r}^2(\omega, \xi^{\pm}) = 0, \quad (2.83)$$

with  $x^{\pm} - q^1$  to be a fixed number as  $q^n \rightarrow \infty$ . Using the above Fourier transforms we end up with the following expressions,

$$\begin{aligned} k_0^n(\lambda) &= \exp\left\{2 \int_0^{\infty} \frac{d\omega}{\omega} \sinh(2i\omega\lambda) \frac{\sinh(\frac{3\omega}{2})e^{-\frac{\omega}{2}}}{\sinh(2\omega)}\right\}, \\ k_1^n(\lambda, \tilde{\xi}_n^{\pm}) &= \exp\left\{-\int_0^{\infty} \frac{d\omega}{\omega} \sinh(2i\omega\lambda) \frac{e^{-2\tilde{\xi}_n^{\pm}\omega}}{\cosh(\omega)}\right\}, \end{aligned} \quad (2.84)$$

where  $2\tilde{\xi}_1^\pm = 2\xi^\pm - 2s^1 + 1$ ,  $2\tilde{\xi}_2^\pm \rightarrow \infty$ , are the renormalized boundary parameters,  $k_0^n(\lambda)$  and  $k_1^n(\lambda)$  are the  $\xi$  independent and the  $\xi$  dependent part, respectively of the usual  $XXX$  ( $SU(2)$ ) reflection matrix (see e.g. [37]). Exactly as in the case of the spin  $s$  open chain, we consider the diagonal  $K$  matrix, therefore we need to determine the other element of each boundary matrix. We exploit the “duality” symmetry [38] of the transfer matrix for  $\xi^\pm \rightarrow -\xi^\pm$ , and we find that the other diagonal element is given by

$$\beta_\pm^n(\lambda, \tilde{\xi}_n^\pm) = e_{2\tilde{\xi}_n^\pm - 1}(\lambda) \alpha_\pm^n(\lambda, \tilde{\xi}_n^\pm). \quad (2.85)$$

The ratio  $\frac{\beta_\pm^n(\lambda, \tilde{\xi}_n^\pm)}{\alpha_\pm^n(\lambda, \tilde{\xi}_n^\pm)}$  is the same as in the  $XXX$  model but with a renormalized boundary parameter. We observe for the alternating chain as well as in the spin  $s$  chain that only one free boundary parameter  $\tilde{\xi}_1^\pm$  ( $\tilde{\xi}_2^\pm \rightarrow \infty$ ) survives.

Two copies of the rational ( $XXX$ ) reflection matrix were computed, one for each excitation. Therefore, we conclude that the boundary  $S$  matrix should be of the structure,

$$S(\lambda) = S_{SU(2)}(\lambda, \tilde{\xi}_1^\pm) \otimes S_{SU(2)}(\lambda, \tilde{\xi}_2^\pm). \quad (2.86)$$

We assume that this matrix should also coincide with the one of the  $PCM_1$ . A calculation of the boundary  $S$  matrix from the field theory point of view would probably confirm our results. We have to mention that there have been some studies for the  $SU(2)$   $PCM$  with  $WZW$  term with boundaries, [43] but mainly in the context of quantum impurity (Kondo) problem. In particular, in [43] the authors considered dynamical boundaries, i.e. they considered quantum impurities at the boundaries, and they derived the corresponding reflection matrices.

### 3 Discussion

The spin  $s$  and the alternating  $s^1, s^2$  ( $s^1 - s^2 = \frac{1}{2}$ ) chain were explored. For both models the Bethe ansatz equations were derived using fusion, and the exact boundary  $S$  matrices were computed. We were particularly interested in the case that  $s, s^i \rightarrow \infty$ . More specifically, for the spin  $s \rightarrow \infty$  chain the boundary scattering amplitudes were simply combined to give the boundary  $S$  matrix of the form  $f(\lambda) S_{SU(2)}(\lambda, \tilde{\xi}_1^\pm) \otimes S_{SU(2)}(\lambda, \tilde{\xi}_2^\pm)$ .

For the alternating spin chain two different types of excitations exist:  $\frac{1}{2}$  and 0 spin respectively. The boundary scattering for each excitation was studied and the corresponding reflection matrices were derived. Two copies of the  $XXX$  boundary  $S$  matrix were computed ( $s^i \rightarrow \infty$ ), and the boundary  $S$  matrix was given as a tensor product of two rational boundary  $S$  matrices. This matrix is also expected to coincide with the one of the  $SU(2)$   $PCM_1$ . Note that we could end up with the same result if

we started from the anisotropic spin chains and then take the isotropic and  $s, s^i \rightarrow \infty$  limits.

It would be interesting to also consider dynamical  $K$  matrices [43] in order to construct the open spin chain and then to study the reflection of the particle-like excitations with the dynamical boundary. Another interesting aspect would be the study of the thermodynamics of the alternating spin chain via the TBA. The main purpose would be the derivation of the central charge (see e.g. [27], [28]), of the model for both bulk and boundary cases. We hope to report on these issues in a future publication.

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## References

- [1] H. Bethe, Z. Phys. **71** (1931) 205.
- [2] L.D. Faddeev and L.A. Takhtajan, Russ. Math. Surv. **34**, 11 (1979); L.D. Faddeev and L.A. Takhtajan, J. Sov. Math. **24** (1984) 241.
- [3] L.D. Faddeev and L.A. Takhtajan, Phys. Lett. **85A** (1981) 375.
- [4] N.Yu. Reshetikhin, Nucl. Phys. **B251** (1985) 565.
- [5] I. Affleck, *In fields strings and critical phenomena*, Les Houches Lectures, (Elsevier 1990).
- [6] N.Yu. Reshetikhin. J. Phys. **A24** (1991) 3299.
- [7] I.V. Cherednik, Theor. Math. Phys. **61** (1984) 977.
- [8] E.K. Sklyanin, J. Phys. **A21** (1988) 2375; P.P. Kulish and E.K. Sklyanin, J. Phys. **A24** (1991) L435.
- [9] S. Ghoshal and A. B. Zamolodchikov, Int. J. Mod. Phys. **A9** (1994) 3841; **A9** (1994) 4353.
- [10] H.J. de Vega and A. González-Ruiz, Mod. Phys. Lett. **A9** (1994) 2207; H.J. de Vega and A. González-Ruiz, J. Phys. **A26** (1993) L519.

- [11] R.J. Baxter, Ann. Phys. **70** (1972) 193; J. Stat. Phys. *8* (1973) 25; *Exactly Solved Models in Statistical Mechanics* (Academic Press, 1982).
- [12] V.E. Korepin, Theor. Math. Phys. **76** (1980) 165; V.E. Korepin, G. Izergin and N.M. Bogoliubov, *Quantum Inverse Scattering Method, Correlation Functions and Algebraic Bethe Ansatz* (Cambridge University Press, 1993).
- [13] A.Kirillov and N.Reshetikhin, J. Sov. Math **35** (1986) 2621; A.Kirillov and N.Reshetikhin, J. Phys. **A20** (1987) 1565.
- [14] L.A. Takhtajan, Phys. Lett. **A87** (1982) 479.
- [15] A.Polyakov and P.Wiegmann Phys. Lett., **B131** (1983) 121.
- [16] P.Wiegmann Phys. Lett. **B142** (1984) 173.
- [17] L.D. Faddev and N.Yu. Reshetikhin, Ann. Phys. **167**(1986) 167.
- [18] H.Babujian, Nucl. Phys. **B215** (1983) 317.
- [19] G.Japaridze, A.Nersessian and P.Wiegmann, Nucl. Phys. **B230** (1984) 511.
- [20] H.Babujian and A.Tsvelik, Nucl. Phys. **B265** (1986) 24.
- [21] A.Zamolodchikov and Al.Zamolodchikov, Nucl. Phys. **B379** (1992) 602.
- [22] D. Bernard, Phys. Lett. **B279** (1992) 78.
- [23] A.Zamolodchikov and Al.Zamolodchikov, Ann. Phys. **120** (1979) 253.
- [24] V.V. Bazhanon and N.Yu. Reshetikhin, Int. J. Mod. Phys. **A4** (1989) 115.
- [25] C. Ahn, D. Bernard and A. Leclair, Nucl. Phys **B346** (1990) 409; D. Bernard, A. Leclair, Phys. Lett. **B247** (1990) 309.
- [26] H.J. de Vega and F. Woyanorovich, J. Phys. **A25** (1992) 4499.
- [27] S.R. Aladim and M.J. Martins, J. Phys. **A26** (1993) 7287.
- [28] H.J. de Vega, L. Mezincescu and R.I. Nepomechie, Phys. Rev. **B49** (1994) 13223.
- [29] H.J de Vega, L. Mezincescu and R.I. Nepomechie, Int. J. Mod. Phys. **B8** (1994) 3473.
- [30] B.D. Doerfel and S. Meisner, J. Phys. **A30** (1996) 6471.
- [31] A. Doikou and A. Babichenko, Phys. Lett **B515** (2001) 220.
- [32] A.Polyakov and P.Wiegmann, Phys. Lett. **B141** (1984) 223.

- [33] L. Mezincescu and R.I. Nepomechie, J. Phys. **A25** (1992) 2533.
- [34] R.I. Nepomechie, *hep-th/0110116*.
- [35] V.Korepin, Theor. Math. Phys **41** (1979) 53.
- [36] N. Andrei and C. Destri, Nucl. Phys. **B131** (1984) 445.
- [37] M. Grisaru, L. Mezincescu and R.I. Nepomechie, J. Phys. **A28** (1995) 1027; A. Doikou, L. Mezincescu and R.I. Nepomechie, J. Phys. **A 30** (1997) L507.
- [38] A. Doikou and R.I. Nepomechie, Nucl. Phys. **B521** (1998) 547; Nucl. Phys. **B530** (1998) 641; Phys. Lett. **B462** (1999) 321.
- [39] N.J. Makcay and B.J. Short, *hep-th/0104212*.
- [40] C.Ahn and W.M. Koo, *hep-th/9708080*; J. Phys. **A29** (1996) 5845.
- [41] R.E. Behrend, P.A. Pearce and D.L. O'Brien, J. Stat. Phys. **84** (1996) 1; R.E. Behrend and P.A. Pearce, J. Phys. **A29** (1996) 7827.
- [42] M.T. Batchelor, V. Fridkin, A. Kuniba and Y.K. Zhou, Phys. Lett **B735** (1996) 266.
- [43] P. Fendley, Phys. Rev Lett. **71** (1993) 2485; J.N. Prata, Phys. Lett. **B438** (1998) 115.